



ELSEVIER

Journal of Geometry and Physics 46 (2003) 394–408

JOURNAL OF
GEOMETRY AND
PHYSICS

www.elsevier.com/locate/jgp

Closed time-like curves in flat Lorentz space–times

Virginie Charette^a, Todd A. Drumm^{b,*}, Dieter Brill^c

^a Department of Mathematics, McMaster University, Hamilton, Ont., Canada

^b Department of Mathematics and Statistics, Swarthmore College, Swarthmore, PA 19081, USA

^c Department of Physics, University of Maryland, College Park, MD 20742, USA

Received 28 June 2002

Abstract

We consider the region of closed time-like curves (CTCs) in three-dimensional flat Lorentz space–times. The interest in this global geometrical feature goes beyond the purely mathematical one. Such space–times are lower-dimensional toy models of sourceless Einstein gravity or cosmology. In three dimensions all such space–times are known: they are quotients of Minkowski space by a suitable group of Poincaré isometries. The presence of CTCs would indicate the possibility of “time machines”, a region of space–time where an object can travel along in time and revisit the same event. Such space–times also provide a testbed for the chronology protection conjecture, which suggests that quantum back reaction would eliminate CTCs. In particular, our interest in this note will be to find the set free of CTCs for $\mathbb{E}/\langle\gamma\rangle$, where \mathbb{E} is modeled on Minkowski space and γ is a Poincaré transformation. We describe the set free of CTCs where γ is hyperbolic, parabolic, and elliptic.

© 2002 Elsevier Science B.V. All rights reserved.

Subj. Class.: General relativity; Differential geometry

Keywords: Closed time-like curves; Flat space–time

1. Introduction

Let \mathbb{E} denote three-dimensional *Minkowski space*. This is an affine space with translations in $\mathbb{R}^{2,1}$, the vector space equipped with the standard indefinite bilinear form $\mathbb{B}(\cdot, \cdot)$ of signature $(2, 1)$. (Since \mathbb{E} is flat and geodesically complete, the reader may identify \mathbb{E} with its set of translations without any major difficulties arising.) We look at flat Lorentz manifolds, which are quotients of an open subset X of \mathbb{E} by a group Γ of affine Lorentzian

* Corresponding author.

E-mail addresses: charette@math.mcmaster.ca (V. Charette), tad@swarthmore.edu (T.A. Drumm), brill@physics.umd.edu (D. Brill).

isometries that acts properly discontinuously on that subset. Such manifolds X/Γ inherit a local causal structure from \mathbb{E} .

Let X/Γ be a flat Lorentz manifold. A *time-like* vector is a vector $v \in \mathbb{R}^{2,1}$ such that $\mathbb{B}(v, v) < 0$. A *time-like curve* in X/Γ is a C^1 path $c : [0, 1] \rightarrow X/\Gamma$ whose tangent vectors are all time-like; we say that c is *closed* if $c(0) = c(1)$. The purpose of this note is to lay the groundwork for understanding regions free of closed time-like curves (CTCs) in a Lorentz space–time.

The *CTC region* of a space–time is the set of all points which lie on some CTC and the *CTC-free region* is the complement of this space. Suppose Γ acts properly discontinuously on some subset $X \subset \mathbb{E}$. Denote by $[p]$ the image of p in X/Γ under projection. We wish to determine the set of all p such that $[p]$ lies in the CTC region of X/Γ .

We first note the following basic lemma.

Lemma 1.1. *Let Γ and $X \subset \mathbb{E}$ be as above. Let $p \in X$ be a point such that $\gamma(p) - p$ is a time-like vector, for some $\gamma \in \Gamma$, and the line segment starting at p and ending at $\gamma(p)$ lies entirely in X . Then $[p] \in X/\Gamma$ lies on a smooth CTC.*

This lemma is well known, but we provide a proof for completeness.

Proof. Let $c : \mathbb{R} \rightarrow \mathbb{E}$ be a continuous and piecewise linear path through each point $\gamma^n(p)$, defined by the following:

$$c(t) = \gamma^{[t]}(p) + (t - [t])(\gamma^{[t]+1}(p) - \gamma^{[t]}(p)),$$

where $[t]$ denotes the largest integer not greater than t . The path c is of the class C^∞ except at the integers. We describe a new path near $t = 0$ and then apply the same procedure at all integer values of t . For some small $\epsilon > 0$, define

$$\tilde{c}(t) = d_{0,\epsilon}(t)(p + t(p - \gamma^{-1}(p))) + u_{0,\epsilon}(t)(p + t(\gamma(p) - p)),$$

where $d_{0,\epsilon}(t)$ and $u_{0,\epsilon}(t)$ are C^∞ functions such that

$$d_{0,\epsilon}(t) = \begin{cases} 1 & \text{for } t \leq 0, \\ 0 & \text{for } t \geq \epsilon, \end{cases} \quad \text{and} \quad u_{0,\epsilon}(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 1 & \text{for } t \geq \epsilon. \end{cases}$$

The path \tilde{c} is C^∞ and agrees with the path c for $t \leq 0$ and $t \geq \epsilon$. The path \tilde{c} can be extended to a C^∞ path through each point $\gamma^n(p)$. We can assume that $\gamma^{n-1}(p) - \gamma^n(n)$ is a future pointing time-like vector. The sum of two future pointing time-like vectors is another future pointing time-like vector. Therefore, \tilde{c} is a smooth time-like curve which goes through each point $\gamma^n(p)$. □

Corollary 1.2. *Suppose Γ acts properly discontinuously on \mathbb{E} . Then $[p]$ is in the CTC region of \mathbb{E}/Γ if and only if $\gamma(p) - p$ is time-like, for some $\gamma \in \Gamma$.*

The vector $\gamma(p) - p$ is called the *displacement vector* for p .

In this note we will mainly consider $\Gamma = \langle \gamma \rangle$, where γ is an affine Lorentzian isometry, that is, an element of the *Poincaré group*. Note that the cyclic group Γ acts properly

discontinuously on all of \mathbb{E} if and only if γ admits no fixed points. We will examine both the fixed point case—in other words, when γ is an element of the *Lorentz group*—and the fixed point-free case.

2. Isometries

Let $\text{Isom}(\mathbb{E})$ denote the group of affine isometries of \mathbb{E} , that is the Poincaré group. Choosing an origin $\mathbf{0}$, we may write the action of an isometry $\gamma \in \text{Isom}(\mathbb{E})$ as

$$\gamma(p) = \mathbf{0} + g(p - \mathbf{0}) + \mathbf{v}, \quad (1)$$

where $g \in \text{O}(2, 1)$ is called its *linear part* and $\mathbf{v} \in \mathbb{R}^{2,1}$ is called its *translational part*. (Typically, we shall omit $\mathbf{0}$, which can be taken to be the point whose coordinates are all zero.)

Denote the Lorentzian inner product on the vector space of translations $\mathbb{R}^{2,1}$ by $\mathbb{B}(\cdot, \cdot)$:

$$\mathbb{B} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = xu + yv - zw.$$

It is invariant under the action due to any element of $\text{O}(2, 1)$.

The union of the sets of time-like vectors and non-zero light-like vectors divides into two connected components, one of which is called *future directed* and one called *past directed*. It is common to choose the connected component where the third coordinate is positive to be the future direction. Such a choice is called a *time-orientation* on \mathbb{E} . We say that a set $X \subset \mathbb{E}$ is *future (resp. past) complete* if given a point $p \in X$ every future (resp. past) directed ray starting at p remains in X .

We will restrict our examination to the identity component $G \subset \text{Isom}(\mathbb{E})$, consisting of orientation and time-orientation preserving isometries. This is a subgroup of $\text{SO}(2, 1)$ which has index 2.

The conjugacy classes of elements of G are identified by their trace. Further, a non-identity element $g \in G$ is called:

- *hyperbolic* if $\text{tr}(g) > 3$,
- *parabolic* if $\text{tr}(g) = 3$ and
- *elliptic* if $\text{tr}(g) < 3$.

We say that an affine transformation is *hyperbolic*, *parabolic*, or *elliptic* if its linear part is hyperbolic, parabolic, or elliptic, respectively. Conjugacy classes of hyperbolic and parabolic affine transformations are determined by the trace of the linear part and what we will call the *signed Lorentzian length*. This invariant, due to Margulis [6,7], measures the Lorentzian length of the closed geodesic determined by the isometry; more on this in the following section.

For a given transformation γ , \mathbb{E} splits into three regions, according to the causal character of the displacement vector $\gamma(p) - p$, for $p \in \mathbb{E}$:

- the *time-like region* for the action of γ on \mathbb{E} :

$$\mathcal{T}(\gamma) = \{p \in \mathbb{E} \mid \mathbb{B}(\gamma(p) - p, \gamma(p) - p) < 0\};$$

- the *light-like region* for the action of γ :

$$\mathcal{L}(\gamma) = \{p \in \mathbb{E} \mid \mathbb{B}(\gamma(p) - p, \gamma(p) - p) = 0\};$$

- the *space-like region* for γ :

$$\mathcal{S}(\gamma) = \{p \in \mathbb{E} \mid \mathbb{B}(\gamma(p) - p, \gamma(p) - p) > 0\}.$$

Suppose $\Gamma \subset \text{Isom}(\mathbb{E})$ acts properly discontinuously on the maximal set $X \subset \mathbb{E}$; we denote by $\mathcal{T}(\Gamma)$ the set of points $p \in X$ such that $[p]$ is in the CTC region of X/Γ . If Γ acts freely and properly discontinuously on \mathbb{E} , then

$$\mathcal{T}(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathcal{T}(\gamma).$$

The region whose quotient is free of CTCs is denoted by $\mathcal{F}(\Gamma)$. If Γ acts freely and properly discontinuously on \mathbb{E} , then

$$\mathcal{F}(\Gamma) = \bigcap_{\gamma \in \Gamma} (\mathcal{S}(\gamma) \cup \mathcal{L}(\gamma)).$$

2.1. Conjugation

An isometry may be conjugated by an element of $\text{GL}(3, \mathbb{R})$, in order to facilitate explicit calculations. For instance, given a hyperbolic $g \in G$, there exists $h \in \text{GL}(3, \mathbb{R})$ such that hgh^{-1} is a diagonal matrix.

Conjugation of an isometry by a linear map corresponds to a change of basis in \mathbb{E} . Conjugation by a pure translation corresponds to changing the origin.

Conjugation of an isometry by elements of the Poincaré group is more restrictive but has real physical meaning. Conjugation by an element of $\text{SO}(2, 1)$ corresponds to changing to an observer in a different inertial reference frame. Conjugation by a translation corresponds to changing to an observer at a different space–time location. The *principle of special relativity* requires the invariance of physical laws under a change of location and inertial reference.

The relevant invariants in our Lorentzian space–times, such as the Lorentzian inner product and signed Lorentzian length, are invariant after conjugation by an element of the Poincaré group. In particular, even though the form of (1) depends on the choice of origin, the signed Lorentzian length does not depend on the choice of origin.

3. Hyperbolic transformations

Let $\gamma \in \text{Isom}(\mathbb{E})$ be a hyperbolic transformation with linear part g . Then g admits three eigenvectors, $\mathbf{x}^-(g)$, $\mathbf{x}^0(g)$, $\mathbf{x}^+(g)$ with eigenvalues $\lambda(g) < 1 < \lambda(g)^{-1}$, respectively. The eigenvectors $\mathbf{x}^\pm(g)$ are null and can be chosen so that they are future directed

and $\mathbb{B}(\mathbf{x}^-(g), \mathbf{x}^+(g)) = -1$. Then we may choose $\mathbf{x}^0(g)$ to be the unique 1-eigenvector satisfying:

- it is unit-space-like, i.e. $\mathbb{B}(\mathbf{x}^0(g), \mathbf{x}^0(g)) = 1$;
- $\{\mathbf{x}^-(g), \mathbf{x}^+(g), \mathbf{x}^0(g)\}$ is a right-handed basis for $\mathbb{R}^{2,1}$.

The signed Lorentzian length of γ is

$$\alpha(\gamma) = \mathbb{B}(\gamma(p) - p, \mathbf{x}^0(g)),$$

where $p \in \mathbb{E}$ is arbitrary (i.e. $\alpha(\gamma)$ does not depend on the choice of p). Every hyperbolic transformation γ gives rise to a unique invariant line $C_\gamma \subset \mathbb{E}$, which is parallel to $\mathbf{x}^0(g)$. If $q \in C_\gamma$:

$$\gamma(q) = q + \alpha(\gamma)\mathbf{x}^0(g),$$

thus $\alpha(\gamma)$ measures the signed Lorentzian length of $C_\gamma/\langle\gamma\rangle$. The curve $C_\gamma/\langle\gamma\rangle$ is called the *unique closed geodesic* in $\mathbb{E}/\langle\gamma\rangle$, that is $C_\gamma/\langle\gamma\rangle$ is the image of a simple closed curve $c[0, 1] \rightarrow \mathbb{E}$ such that $c'(t)$ is constant on $[0, 1]$, where $c'(0)$ is thought of as a right-hand limit and $c'(1)$ is thought of as a left-hand limit.

As trace identifies conjugacy classes of elements in $SO(2, 1)$, the trace of the linear part and the value of α identify conjugacy classes of hyperbolic isometries in $\text{Isom}(\mathbb{E})$. Conjugate γ by the change of basis matrix $[\mathbf{x}^-(g)\mathbf{x}^+(g)\mathbf{x}^0(g)]$ (an element of $GL(3, \mathbb{R})$, not $SO(2, 1)$), so that its linear part is diagonal. Next, conjugate by an appropriate translation, so that the origin lies on C_γ .

Write the point $p \in \mathbb{E}$ in terms of the new basis and origin:

$$p = p_-\mathbf{x}^-(g) + p_+\mathbf{x}^+(g) + p_0\mathbf{x}^0(g).$$

We may write the conjugate transformation as follows:

$$\begin{bmatrix} p_- \\ p_+ \\ p_0 \end{bmatrix} \mapsto \begin{bmatrix} \lambda(g) & 0 & 0 \\ 0 & \lambda(g)^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_- \\ p_+ \\ p_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \alpha(\gamma) \end{bmatrix} = \begin{bmatrix} \lambda(g)p_- \\ \lambda(g)^{-1}p_+ \\ p_0 + \alpha(\gamma) \end{bmatrix}.$$

The causal character of the displacement vector is determined by the following calculation:

$$\mathbb{B}(\gamma(p) - p, \gamma(p) - p) = 2\mathbb{B}(\mathbf{x}^+(g), \mathbf{x}^-(g))(\lambda(g) - 1)(\lambda(g)^{-1} - 1)p_-p_+ + \alpha(\gamma)^2.$$

Thus points in $\mathcal{T}(\gamma)$, i.e. points projected onto CTCs, satisfy the following inequality:

$$p_-p_+ < \frac{-\alpha(\gamma)^2}{2(1 - \lambda(g))(\lambda(g)^{-1} - 1)}. \tag{2}$$

Observe that the right-hand side is negative.

3.1. Hyperbolic transformations with fixed points

Suppose that γ admits a fixed point; equivalently, C_γ is pointwise fixed and $\alpha(\gamma) = 0$. Set,

$$\mathcal{B}_\gamma = \{p_-\mathbf{x}^-(\gamma) + p_+\mathbf{x}^+(\gamma) + p_0\mathbf{x}^0(\gamma) | p_-p_+ < 0\}.$$

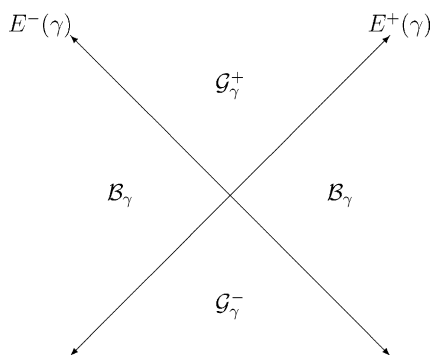


Fig. 1. A cross-section for hyperbolic transformations.

Then p solves (2) if and only if $p \in \mathcal{B}_\gamma$. This region divides into two connected components, bounded by $\mathcal{L}(\gamma)$, which in turn is composed of the so-called *unstable/stable* planes $E^\pm(\gamma)$: these are the planes containing C_γ and parallel to $\mathbf{x}^\pm(g)$.

The closure of the two remaining components of \mathbb{E} :

$$\mathcal{G}_\gamma^+ = \{p_- \mathbf{x}^-(\gamma) + p_+ \mathbf{x}^+(\gamma) + p_0 \mathbf{x}^0(\gamma) \mid p_-, p_+ \geq 0\},$$

$$\mathcal{G}_\gamma^- = \{p_- \mathbf{x}^-(\gamma) + p_+ \mathbf{x}^+(\gamma) + p_0 \mathbf{x}^0(\gamma) \mid p_-, p_+ \leq 0\},$$

form $\mathcal{F}(\gamma)$, see Fig. 1. The set \mathcal{G}_γ^+ is future complete.

Note that $\mathcal{F}(\gamma) = \mathcal{F}(\gamma^n)$, for every n , since $\mathbf{x}^\pm(\gamma^n) = \mathbf{x}^\pm(\gamma)$. Thus we have the following proposition.

Proposition 3.1. *Let γ be a hyperbolic transformation with fixed points. Then*

$$\mathcal{F}(\langle \gamma \rangle) = \mathcal{G}_\gamma^+ \cup \mathcal{G}_\gamma^- \setminus C_\gamma,$$

and

$$\mathcal{T}(\langle \gamma \rangle) = \mathcal{B}_\gamma.$$

The interior of \mathcal{G}_γ^+ is a maximal connected open γ -invariant CTC-free region of \mathbb{E} on which $\langle \gamma \rangle$ acts properly discontinuously. The quotient $\mathcal{G}_\gamma^+ / \langle \gamma \rangle$ is called *Misner space* and is a Lorentz space–time which is future complete and diffeomorphic to $\mathbb{R}^2 \times S^1$.

The interior of \mathcal{G}_γ^- yields an analogous, past complete space–time.

3.2. Hyperbolic transformations without fixed points

Define $\mathcal{B}_\gamma, \mathcal{G}_\gamma^+, \mathcal{G}_\gamma^-$ as above. Solutions to (2) now consist of two components which are strictly contained in \mathcal{B}_γ . They are bounded by hyperbolic sheets forming $\mathcal{L}(\gamma)$, which are asymptotic to the planes $E^\pm(\gamma)$.

Theorem 3.2. Let $\gamma \in \text{Isom}(\mathbb{E})$ be a hyperbolic isometry without fixed points; then

$$\mathcal{T}(\langle \gamma \rangle) = \mathcal{B}_\gamma,$$

and the CTC region of $\mathbb{E}/\langle \gamma \rangle$ is $\mathcal{B}_\gamma/\langle \gamma \rangle$. Furthermore,

$$\mathcal{F}(\langle \gamma \rangle) = \mathcal{G}_\gamma^+ \cup \mathcal{G}_\gamma^-.$$

The regions \mathcal{G}_γ^+ and \mathcal{G}_γ^- are future complete and past complete, respectively. All closed curves in $\mathcal{G}_\gamma^+/\langle \gamma \rangle$ and $\mathcal{G}_\gamma^-/\langle \gamma \rangle$ are space-like.

Proof. Note that the linear part of γ^n is g^n so $\lambda(g^n) = (\lambda(g))^n$, but $\alpha(\gamma^n) = n\alpha(\gamma)$. Consider the right-hand side of (2). We find

$$\lim_{n \rightarrow \infty} \frac{-\alpha(\gamma^n)^2}{2(1 - \lambda(g^n))(\lambda(g^n)^{-1} - 1)} = 0.$$

Therefore, the hyperbolic sheets $\mathcal{L}(\gamma^n)$ approach $E^\pm(\gamma)$, since $E^\pm(\gamma^n) = E^\pm(\gamma)$. □

In contrast to hyperbolic transformations with fixed points, the maximal connected open subset of \mathbb{E} on which $\langle \gamma \rangle$ acts properly discontinuously is all of \mathbb{E} . The quotient $\mathbb{E}/\langle \gamma \rangle$ is a Lorentz space–time which is future complete, past complete, and diffeomorphic to $\mathbb{R}^2 \times S^1$, which Grant [5] identified as representing the complement of two straight moving cosmic strings which do not intersect.

Now that we understand the CTC regions for a group generated by a single hyperbolic transformation, we can start to look at more complicated groups. Here is an interesting example, where the CTC region consists of all of \mathbb{E} .

Example 2.3 ($\mathcal{F}(\Gamma) = \emptyset$). Suppose that γ_1 and γ_2 are hyperbolic transformations such that:

- the invariant lines $C_{\gamma_1}, C_{\gamma_2}$ are distinct,
- the linear parts admit the same eigenvectors, and
- the group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ acts freely and properly discontinuously on \mathbb{E} .

Note that there are such groups. One such example is the group generated by the transformations

$$\gamma_1(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{\sqrt{5}}{2} \\ 0 & -\frac{\sqrt{5}}{2} & \frac{3}{2} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and}$$

$$\gamma_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{2} & -\frac{3\sqrt{5}}{2} \\ 0 & -\frac{3\sqrt{5}}{2} & \frac{7}{2} \end{bmatrix} x + \begin{bmatrix} 2 \\ \frac{1}{\sqrt{5}} \\ 1 \end{bmatrix}.$$

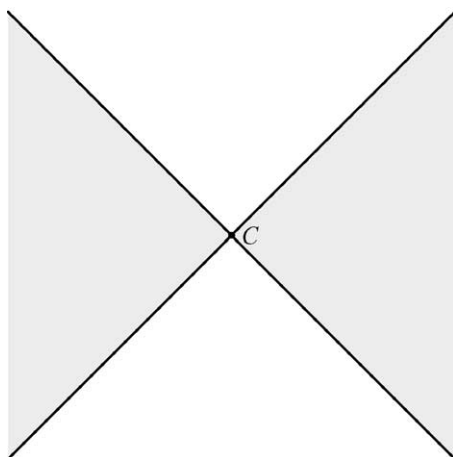


Fig. 2. A cross-section of \mathcal{B}_{γ_1} ; $C = C_{\gamma_1}$.

These transformations admit distinct invariant lines which are both parallel to the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(In fact, $\mathbb{E}/\langle \gamma_1, \gamma_2 \rangle$ is a 3-torus. There is a normal subgroup $\cong \mathbb{Z}^2$ of pure translations and the linear parts preserve this \mathbb{Z}^2 -lattice.)

To study $\mathcal{T}(I)$, it is enough to look at a single cross-section parallel to

$$\langle \mathbf{x}^-(g_1), \mathbf{x}^+(g_1) \rangle = \langle \mathbf{x}^-(g_2), \mathbf{x}^+(g_2) \rangle.$$

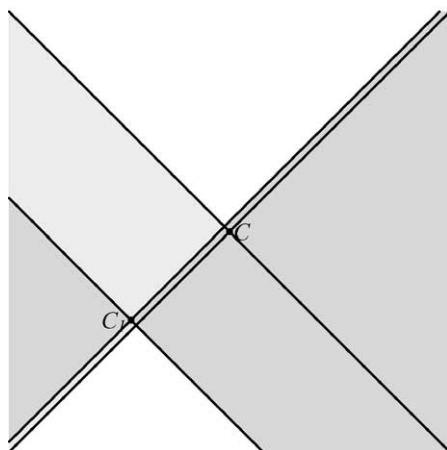


Fig. 3. A cross-section of \mathcal{B}_{γ_2} and \mathcal{B}_{γ_1} ; $C_1 = C_{\gamma_2}$.

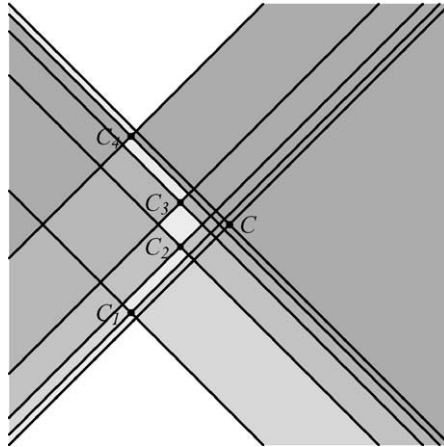


Fig. 4. A cross-section of $\mathcal{B}_{\gamma_1^n \gamma_2 \gamma_1^{-i}}$, $0 \leq i \leq 3$, and \mathcal{B}_{γ_1} ; $C_i = C_{\gamma_1^{i-1} \gamma_2 \gamma_1^{-i+1}}$.

Conjugate invariant lines are obtained as follows:

$$C_{\gamma_1 \gamma_j \gamma_i^{-1}} = \gamma_i C_{\gamma_j}.$$

Suppose that $q \in \mathcal{G}_{\gamma_1}^+ \cup \mathcal{G}_{\gamma_2}^+$. The point representing the invariant line $C_{\gamma_1^n \gamma_2 \gamma_1^{-n}}$ approaches the asymptote representing either $E^+(\gamma_1)$ or $E^-(\gamma_1)$ as seen in Figs. 2–4. So in our example, every point q lies in $\mathcal{B}_{\gamma_1^n \gamma_2 \gamma_1^{-n}}$ for a large enough n . The other cases are treated in the same manner, showing that every q lies in \mathcal{B}_γ , for some $\gamma \in \Gamma$.

Thus, the CTC region of \mathbb{E}/Γ is the entire space.

4. Parabolic transformations

In the linear group G , all parabolic transformations are conjugate to each other. The conjugacy classes of affine parabolic transformations in $\text{Isom}(\mathbb{E})$ are described in [1]. In particular, the invariant α , which is defined for hyperbolic transformations only, can be generalized to parabolic transformations.

4.1. Parabolic transformations with fixed points

Let $\rho \in \text{Isom}(\mathbb{E})$ be a parabolic isometry with a fixed point. Equivalently, ρ admits a pointwise fixed light-like line, which is parallel to the unique (fixed) eigendirection of ρ 's linear part.

For simplicity, identify ρ with ρ 's linear part in G —this may be achieved by choosing an origin 0 in the fixed point set of ρ . (We will omit 0 in the rest of this discussion.) Thus ρ admits a fixed eigendirection of light-like vectors. Choose one such vector which is future pointing and call it $x^0(\rho)$.

Let $\mathbf{x}^0(\rho)^\perp$ denote its Lorentz-orthogonal plane in $\mathbb{R}^{2,1}$, that is, the set of vectors whose Lorentz inner product with $\mathbf{x}^0(\rho)$ vanishes. Note that $\mathbf{x}^0(\rho)^\perp$ is tangent to the light cone and contains $\mathbb{R}\mathbf{x}^0(\rho)$, the line spanned by $\mathbf{x}^0(\rho)$. Every vector in $\mathbf{x}^0(\rho)^\perp$ is either light-like (if and only if it is parallel to $\mathbf{x}^0(\rho)$) or space-like; in particular, $\mathbf{x}^0(\rho)^\perp$ contains no time-like vectors.

Similarly, to the hyperbolic case, let $\mathbf{x}^1, \mathbf{x}^2$ be a pair of vectors such that:

- \mathbf{x}^1 lies in $\mathbf{x}^0(\rho)^\perp$ and is unit-space-like;
- \mathbf{x}^2 is light-like, future pointing and Lorentz-orthogonal to \mathbf{x}^1 , but not parallel to $\mathbf{x}^0(\rho)$;
- $\{\mathbf{x}^0(\rho), \mathbf{x}^1, \mathbf{x}^2\}$ is a positively oriented basis.

Then relative to this basis, every power of ρ can be written as an upper-triangular matrix with 1's on the diagonal:

$$\rho^n = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix}.$$

In particular, for every $p \in \mathbb{E}$ each displacement vector $\rho^n(p) - p$ lies in $\mathbf{x}^0(\rho)^\perp$. (More generally, if $\gamma \in \text{Isom}(\mathbb{E})$ admits fixed points, then the displacement vectors lie in $\mathbf{x}^0(\rho)^\perp$. We will use this fact again in the elliptic case.) Thus the displacement vector is never time-like. The displacement vector is light-like if and only if $p \in \mathbf{x}^0(\rho)^\perp$ and, of course, vanishes if and only if p is fixed by ρ .

Thus,

$$\mathcal{S}(\rho) = \mathcal{S}(\rho^n) = \mathbb{E} - \mathbf{x}^0(\rho)^\perp, \quad \mathcal{L}(\rho) = \mathcal{L}(\rho^n) = \mathbf{x}^0(\rho)^\perp.$$

We define

$$\mathbb{E}_\rho = \mathbb{E} - \mathbb{R}\mathbf{x}^0(\rho),$$

and we have shown the following proposition.

Proposition 4.1. *Let ρ be a parabolic isometry with fixed points. Then*

$$\mathcal{F}(\langle \rho \rangle) = \mathbb{E}_\rho \quad \text{and} \quad \mathcal{T}(\langle \gamma \rangle) = \emptyset.$$

That is, the CTC region of $\mathbb{E}_\rho / \langle \rho \rangle$ is empty. The space-like region $\mathcal{S}(\rho)$ divides into two components separated by the light-like region $\mathcal{L}(\rho) = \mathbf{x}^0(\rho)^\perp - \mathbb{R}\mathbf{x}^0(\rho)$.

4.2. Parabolic transformations with no fixed points

The situation changes dramatically for parabolic transformations without fixed points.

Recall that all linear parabolics are conjugate to each other. Choose ρ such that

$$\{\mathbf{x}^0(\rho), \mathbf{x}^1, \mathbf{x}^2\} = \left\{ \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

Choose an origin 0 and denote the coordinates of a point p in the $\{\mathbf{x}^0(\rho), \mathbf{x}^1, \mathbf{x}^2\}$ basis as follows:

$$p = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}.$$

We may further conjugate the parabolic isometry so that its translational part is parallel to \mathbf{x}^2 ; then the transformation may be written as

$$\rho_\tau(p) = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix}. \tag{3}$$

The inner product in this basis is given by

$$\mathbb{B}(p, q) = -p_0q_2 + p_1q_1 - p_2q_0,$$

where q_0, q_1, q_2 are the coordinates of q in the new basis. A point p is in $\mathcal{T}(\rho_\tau)$ if

$$p_2^2 - \tau(\sqrt{2}p_1 + p_2) < 0.$$

More explicitly,

$$p_1 > \frac{p_2^2 - \tau p_2}{\sqrt{2}\tau} \quad \text{if } \tau > 0, \quad p_1 < \frac{p_2^2 - \tau p_2}{\sqrt{2}\tau} \quad \text{if } \tau < 0. \tag{4}$$

We see that $\mathcal{L}(\rho_\tau)$ is a parabolic sheet bounding $\mathcal{T}(\rho_\tau)$. Let us examine the case where $\tau > 0$. Suppose that $p \in \mathcal{L}(\rho_\tau) \cup \mathcal{T}(\rho_\tau)$. The set of future pointing, non-space-like vectors is convex. Therefore,

$$\rho_\tau^{n+1}(p) - p = (\rho_\tau^{n+1}(p) - \rho_\tau(p)) + (\rho_\tau(p) - p) = g(\rho_\tau^n(p) - p) + (\rho_\tau(p) - p)$$

is a future pointing time-like vector, since each term is either a future pointing light-like vector or a future pointing time-like vector and all vectors are not parallel.

The same statement holds when $\tau < 0$, substituting the term “past” for “future”.

Lemma 4.2. *Let $\tau \neq 0$; then $\mathcal{T}(\rho_\tau^n) \subset \mathcal{T}(\rho_\tau^{n+1})$ for all $n \geq 1$.*

We will now show by direct calculation that the parabolic sheet $\mathcal{L}(\rho_\tau^n)$ is described by the equation

$$p_1 = \frac{p_2^2 - \tau p_2}{\sqrt{2}\tau} - \phi(n)\tau, \tag{5}$$

where $\phi(n)$ increases as fast as n^2 . Notice that this sheet is just a translate of $\mathcal{L}(\rho_\tau)$ in the x^1 direction by the amount $-\phi(n)\tau$.

We can show the following by induction:

$$\rho_\tau^n(p) = \begin{bmatrix} n\sqrt{2}p_1 + n^2 p_2 + \tau \sum_{i=1}^{n-1} i^2 \\ n\sqrt{2}p_2 + \sqrt{2}\tau \sum_{i=1}^{n-1} i \\ n\tau \end{bmatrix}. \tag{6}$$

We get an equation describing $\mathcal{L}(\rho_\tau^n)$ by solving

$$\mathbb{B}(\rho_\tau^n(p) - p, \rho_\tau^n(p) - p) = 0,$$

and we obtain

$$n^2\sqrt{2}\tau p_1 = n^2 p_2^2 + n\tau p_2 \left(-n^2 + 2 \sum_{i=1}^{n-1} i \right) - \tau^2 \left(n \sum_{i=1}^{n-1} i^2 - \left(\sum_{i=1}^{n-1} i \right)^2 \right).$$

We note that $-n^2 + 2 \sum_{i=1}^{n-1} i = -n$ and that $6 \sum_{i=1}^{n-1} i^2 = n(n-1)(2n-1)$, so the following holds:

$$p_1 = \frac{p_2^2 - \tau p_2}{\sqrt{2}\tau} - \frac{\tau(n^2 - 1)}{12\sqrt{2}}.$$

Theorem 4.3. *Let $\rho \in \text{Isom}(\mathbb{E})$ be a parabolic isometry without fixed points. Then*

$$\mathcal{T}(\langle \gamma \rangle) = \mathbb{E},$$

and the CTC region of $\mathbb{E}/\langle \gamma \rangle$ is the entire space.

Example 4.4. Suppose $\{g_n\}_{n \geq 0} \subset G$ is a sequence of hyperbolic isometries with a common fixed point 0 , converging to a parabolic element in G . This happens, for instance, by letting $g_n = h^n g h^{-n}$, where g is hyperbolic and h is an arbitrary element of G . Then,

$$\mathcal{T}(g_n) \rightarrow \mathcal{T}(\rho) \quad \text{as } g_n \rightarrow \rho.$$

Indeed, as $g_n \rightarrow \rho$, the stable and unstable planes of g_n both approach the plane $x^0(\rho)^\perp$. (Recall that $x^0(\rho)^\perp$ is tangent to the light cone at the origin and contains $\mathbb{R}x^0(\rho)$.) Thus $\mathcal{T}(g_n)$ approaches the empty set.

Now consider a sequence of hyperbolic transformations $\{\gamma_n\}_{n \geq 0}$ approaching a parabolic transformation ρ , without fixed points. The regions $\mathcal{T}(\gamma_n)$ still approach the empty set. However, $\mathcal{T}(\rho) = \mathbb{E}$.

5. Elliptic transformations

In G , all elliptic elements are conjugate to an element of the form

$$\psi_\theta = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

The signed Lorentzian length invariant α cannot be generalized in any coherent manner to elliptic elements. We will first consider elliptic transformations with fixed points and then without fixed points.

5.1. Elliptic transformations with fixed points

After choosing an origin 0 in the fixed point set, we let

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so that the transformation can be written as ψ_θ . A fixed eigenvector for this transformation is

$$\mathbf{x}^0(\psi_\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the fixed point set for ψ_θ consists of the line $\mathbb{R}\mathbf{x}^0(\psi_\theta)$.

If θ is a rational multiple of 2π , then the group $\langle \psi_\theta \rangle$ acts freely and properly discontinuously on the complement of its fixed point set

$$\mathbb{E}_{\psi_\theta} = \mathbb{E} - \mathbb{R}\mathbf{x}^0(\psi_\theta).$$

Recall that since $\mathbf{x}^0(\psi_\theta)$ is fixed by ψ_θ , every displacement vector lies in the Lorentz-orthogonal plane of $\mathbf{x}^0(\psi_\theta)$. This is the xy -plane, which is space-like. Thus we have the following proposition.

Proposition 5.1. *Suppose ψ is an elliptic isometry with fixed points, and that ψ is a rotation of a rational multiple of 2π about its line of fixed points. Then,*

$$\mathcal{F}(\langle \psi_\theta \rangle) = \mathbb{E}_{\psi_\theta} \quad \text{and} \quad \mathcal{T}(\langle \psi_\theta \rangle) = \emptyset.$$

That is, the CTC region of $\mathbb{E}_{\psi_\theta} / \langle \psi_\theta \rangle$ is empty.

Spaces of the type $\mathbb{E}_{\psi_\theta} / \langle \psi_\theta \rangle$ can be identified with special cases of space–times that represent (spinless) particles in $(2+1)$ -dimensional gravity. Such spaces have been described

by Deser et al. [3] (see also [9] and references cited there). The direct metric product of such a space with a space-like line is known as a cosmic string.

5.2. Elliptic transformations without fixed points

As in the parabolic case, the situation changes dramatically for elliptic transformations without fixed points. Every elliptic transformation is conjugate to a transformation

$$\psi_{\theta,t}(p) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \tag{8}$$

which we will write as ψ . The group $\langle \psi \rangle$ acts properly discontinuously on all of \mathbb{E} .

Theorem 5.2. *Let $\psi \in \text{Isom}(\mathbb{E})$ be an elliptic isometry without fixed points. Then,*

$$\mathcal{T}(\langle \psi \rangle) = \mathbb{E},$$

and the CTC region of $\mathbb{E}/\langle \psi \rangle$ is the entire space.

Proof. Note that

$$\psi^k = \begin{bmatrix} \cos k\theta & \sin k\theta & 0 \\ -\sin k\theta & \cos k\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ kt \end{bmatrix},$$

so that the length (equivalent to the underlying topology) of the projection of $\psi(p) - p$ onto the (x, y) -plane is bounded by $2\|x^2 + y^2\|$. However, the projection of $\psi(p) - p$ onto the z -axis is unbounded as $k \rightarrow \infty$. Thus, for a sufficiently large power k , depending on the distance from p to the z -axis, the vector $\psi^k(p) - p$ is time-like. \square

6. Future directions (pun intended)

In a future note, we will look for CTCs in X/Γ , where $X \subset \mathbb{E}$ and Γ is more complicated. Some examples are given below.

For the case $X = \mathbb{E}$ and free Γ , we call \mathbb{E}/Γ a Margulis space-time. Theorem 4.3 has an interesting and immediate consequence for Margulis space-times. In [2], Margulis space-times with non-cyclic free fundamental groups containing parabolic transformations were constructed. Parabolic transformations were shown to be very much like hyperbolic transformations for questions concerning proper actions of a group on \mathbb{E} in [2] (and [1] for that matter). But we see here that for questions concerning CTCs, the difference between hyperbolic and parabolic transformations is tremendous.

We will also be keenly interested in surface groups, groups isomorphic to the fundamental group of a closed surface. As shown in [4,8] these groups do not act properly discontinuously

on \mathbb{E} . However, Mess [8] showed that surface groups can act properly discontinuously on some subset $X \subset \mathbb{E}$.

References

- [1] V. Charette, T. Drumm, Signed Lorentzian displacement for parabolic transformations, submitted for publication.
- [2] T. Drumm, Linear holonomy of Margulis space–times, *J. Diff. Geom.* 38 (1993) 679–691.
- [3] S. Deser, R. Jackiw, G. 't Hooft, Three-dimensional Einstein gravity: dynamics of flat space, *Ann. Phys.* 152 (1984) 220–235;
S. Deser, R. Jackiw, G. 't Hooft, Physical cosmic strings do not generate closed timelike curves, *Phys. Rev. Lett.* 68 (1992) 267–269.
- [4] W. Goldman, G. Margulis, Flat Lorentz 3-manifolds and cocompact Fuchsian groups, *Contemp. Math.* 262 (2000) 135–146.
- [5] J.D.E. Grant, Cosmic strings and chronology protection, *Phys. Rev. D* 47 (1993) 2388–2394.
- [6] G. Margulis, Free properly discontinuous groups of affine transformations, *Dokl. Akad. Nauk SSSR* 272 (1983) 937–940.
- [7] G. Margulis, Complete affine locally flat manifolds with a free fundamental group, *J. Sov. Math.* 134 (1987) 129–134.
- [8] G. Mess, Lorentz space–times of constant curvature, Preprint, 1990.
- [9] M. Welling, One-particle Hilbert space of $(2 + 1)$ -dimensional gravity using non-commuting coordinates, *Nucl. Phys. B* 57 (1997) 346–349. gr-qc/9703057.